Int. J. Solids Structures, 1975, Vol. 11, pp. 1051–1056. Pergamon Press. Printed in Gt. Britain.

RIGOROUS EXPUNCTION OF POISSON'S RATIO FROM THE REISSNER-MEISSNER EQUATIONS†

JAMES G. SIMMONDS
Department of Applied Mathematics and Computer Science, University of Virginia, Charlottesville, VA 22901, U.S.A.

(Received 9 January 1975)

Abstract—Solutions of the two Reissner-Meissner linear equations for the torsionless, axisymetric deformation of elastically isotropic shells of revolution of constant thickness subject to edge conditions and variable normal pressure are compared with the solutions of a simplified version of these equations obtained by neglecting terms containing Poisson's ratio. The relative pointwise differences in the predicted values for the change in the meridional angle and a stress function are shown to be of the same order of magnitude as the inherent errors in classical, first-approximation shell theory. These results do not depend on physical arguments or asymptotic integration techniques, but rather follow from the structure of the Reissner-Meissner equations themselves. The advantage of the simplified equations is that they may be combined into a single comple-valued equation containing no conjugates of the dependent variable.

1. INTRODUCTION

The equations governing the torsionless, axisymmetric deformation of shells of revolution can be expressed as two coupled second order ordinary differential equations—a remarkable reduction considering that in the classical, first-approximation theory of general shells, the field equations are a system of eighth order partial differential equations. Symmetry, of course, makes all differential equations ordinary, and exclusion of torsion reduces the order of the general system by two. An additional reduction in order is achieved by explicit integration of the differential equation of axial force equilibrium. The final reduction in order results from an integration of the analogous compatibility condition.

A special case of this reduction was first achieved in 1912 by H. Reissner[1], who reduced the linear equations for the torsionless, axisymmetric deformation of a spherical shell to two coupled equations for β, the change in the midsurface meridional angle and Q, the transverse shear stress resultant. Shortly thereafter, Meissner[2] extended Reissner's reduction to arbitrary shells of revolution. In 1949, E. Reissner[3] introduced a valuable modification into the Reissner-Meissner equations by replacing the dependent variable Q by rH, where r is the radial distance from the shell axis and H is the horizontal stress resultant, thereby obtaining a set of equations which passed uniformly into the uncoupled equations for the axisymmetric bending and stretching of a plate. (Subsequently, E. Reissner[4] showed that the same choice of dependent variables leads to an analogous and concise reduction of the equations governing the torsionless, *nonlinear* axisymmetric deformation of shells of revolution.)

An additional simplification of the Reissner-Meissner equations, proposed by several writers (for example Naghdi and DeSilva[5], Baker and Cline[6], Steele and Hartung[7], and Clark[8]), is to neglect terms containing the Poisson ratio factors $(1 \pm \nu)$. This step enables the two resulting equations to be combined into a single, complex-valued second order differential equation which, among other things, facilitates the application of asymptotic integration techniques[8].

The neglect of such terms may be justified, even for arbitrary shells, on physical grounds[9, 10]. For shells of revolution, asymptotic integration techniques imply that the influence of such terms is negligible. However, there are drawbacks to these arguments: the physical arguments yield no pointwise estimates of the solution errors induced by dropping the Poisson-ratio terms, while the error estimates supplied by the asymptotic methods may be too pessimistic in certain cases. Moreover to make these latter error estimates rigorous, one must use the properties of the solutions of a comparison differential equation. The analytical nature of these solutions depends critically on the geometry of the mid-surface meridian in the neighborhood of any points on the axis of revolution or where the slope is zero, and may be complicated.

†This research was supported by the National Science Foundation under Grant GP-38197.

The aim of the present paper is to provide a rigorous estimate of the solution errors induced by neglecting Poisson-ratio terms in the Reissner-Meissner equations. We show, for shells subject to prescribed edge forces and rotations and variable normal pressure, that

$$|\beta - \beta_*|, |\psi - \psi_*| \leqq \epsilon^2 N(\eta) U_*^{1/2}, \tag{1.1}$$

where ψ is a nondimensional form of rH and β_* and ψ_* are the solutions of the simplified Reissner-Meissner equations. Here $R\eta$ is meridional arc length and R is some characteristic geometrical dimension of the midsurface; $N(\eta)$ is a function which remains bounded providing the meridian does not meet the axis of revolution in a cusp, ϵ^2 is proportional to h/R, where h is the shell thickness (taken constant) and U_* is the dimensionless strain energy functional associated with the Reissner-Meissner equations, evaluated at the solution of the simplified Reissner-Meissner equations. We note that asymptotic integration techniques yield a formal error estimate that is only $0(\epsilon)$ (see eqn (24) of [8]). However, the best estimates we can expect for the derivative differences $|\beta' - \beta_*'|$ and $|\psi' - \psi_*'|$ are $0(\epsilon)$.

2. THE REISSNER-MEISSNER EQUATIONS

Consider a shell of revolution whose midsurface, referred to a set of right-hand cylindrical coordinates (r, θ, z), is specified in the parametric form $r = r(s)$, $z = z(s)$, $0 \leqq s \leqq L$, where s is arc length. For torsionless, axisymmetric deformation, the linear moment equilibrium and compatibility conditions—the two fundamental equations in the Reissner-Meissner theory—can be expressed as

$$(rM_s)' - M_\theta \cos\phi - rQ = 0 \tag{2.1}$$

$$(r\Gamma_\theta)' - \Gamma_s \cos\phi + \beta \sin\phi = 0. \tag{2.2}$$

Here, M_s and M_θ are stress couples, Γ_s and Γ_θ are meridional and hoop strains, $\cos\phi = r'(s)$, $\sin\phi = z'(s)$ and a prime denotes differentiation with respect to s.

The two equations of force equilibrium involve the stress resultants N_s, N_θ and Q and are satisfied identically by taking

$$rN_s = \Psi \cos\phi + rV \sin\phi \tag{2.3}$$

$$N_\theta = \Psi' + rP \sin\phi \tag{2.4}$$

$$rQ = \Psi \sin\phi - rV \cos\phi, \tag{2.5}$$

where Ψ is a stress function,

$$rV = r(0)V(0) + \int_0^s P(t)r(t)\cos\phi(t)\,dt, \tag{2.6}$$

and the only external surface load is assumed to be a variable normal pressure P. V represents the vertical force per unit length acting along a parallel circle of the midsurface.

We assume that the shell obeys the following elastically isotropic, uncoupled stress-strain relations:

$$M_s = D(K_s + \sigma K_\theta),\ M_\theta = D(K_\theta + \sigma K_s) \tag{2.7}$$

$$\Gamma_\theta = A\ (N_\theta - \nu N_s),\ \Gamma_s = A\ (N_s - \nu N_\theta), \tag{2.8}$$

where the bending strains are related to the meridional angle of rotation by

$$K_s = \beta',\ rK_\theta = \beta\cos\phi. \tag{2.9}$$

It is conventional, but not necessary, to take

$$D = \frac{Eh^3}{12(1-\nu^2)},\ A = \frac{1}{Eh},\ \sigma = \nu, \tag{2.10}$$

where E is Young's modulus.

Inserting (2.3), (2.4) and (2.7)–(2.9) into (2.1) and (2.2), and introducing dimensionless quantities through the relations

$$\begin{aligned} &s = R\eta, \quad L = Rl, \quad r = R\rho, \quad \Psi = \sqrt{(D/A)}\psi \\ &Q = \sqrt{(D/A)}(q/R), \quad V = \sqrt{(D/A)}(v/R), \quad P = \sqrt{(D/A)}(p/R^2) \\ &\epsilon^2\sqrt{(DA)}/R = 0(h/R), \end{aligned} \tag{2.11}$$

we have

$$\epsilon^2\{(\rho\beta' + \sigma\beta \cos\phi)' - (\beta/\rho)\cos^2\phi - \sigma\beta' \cos\phi\} = \rho q \tag{2.12}$$

$$\begin{aligned} &\epsilon^2\{[\rho\psi' + p\rho^2 \sin\phi - \nu(\psi\cos\phi + \rho v \sin\phi)]' \\ &\quad - [(\psi/\rho)\cos\phi + v\sin\phi - \nu(\psi' + p\rho\sin\phi)]\cos\phi\} = -\beta\sin\phi, \end{aligned} \tag{2.13}$$

where now primes denote differentiation with respect to η and

$$\rho q = \Psi \sin\phi - \rho v \cos\phi. \tag{2.14}$$

Noting that (2.6) implies $(\rho v)' = p\rho\cos\phi$, we obtain, after some algebra, the Reissner-Meissner equations

$$\epsilon^2[L\beta + \underline{(1-\sigma)\phi'\sin\phi\beta}] - \rho q = 0 \tag{2.15}$$

$$\epsilon^2[L\Psi + \underline{(1+\nu)\phi'\rho q}] + \beta\sin\phi = \epsilon^2[(\sin\phi - \rho\phi')v\cos\phi - (p\rho^2\sin)'], \tag{2.16}$$

where

$$Lf = \rho[\rho^{-1}(\rho f)']'. \tag{2.17}$$

The simplified Reissner-Meissner equations follow upon neglecting the underlined terms. With the aid of (2.14), they may be expressed as

$$\epsilon^2 L\beta_* - \psi_*\sin\phi = -\rho v\cos\phi \tag{2.18}$$

$$\epsilon^2 L\psi_* + \beta_*\sin\phi = \epsilon^2[(\sin\phi - \rho\phi')v\cos\phi - (p\rho^2\sin\phi)']. \tag{2.19}$$

3. RIGOROUS JUSTIFICATION OF THE SIMPLIFIED REISSNER-MEISSNER EQUATIONS

Clark[8] justified the neglect of the underlined terms in (2.15) and (2.16) on two grounds. First, he remarked that in particular problems β and q are of the same order of magnitude. Hence the neglected terms can be expected to be $0(\epsilon^2)$. And second, he observed that the inclusion of transverse shearing strains results in an order one modification of the underlined term in (2.16) (See eqn. (29) of [11]). Hence the neglect of this term is consistent with the basic assumptions of first-approximation shell theory.

The arguments used by the author[9] and Danielson[10] to justify neglecting similar terms in general shell theory are related to Clark's and assume that in equations resulting from the introduction of stress-strain relations, any term may be neglected which is of the same order and type as any term which would arise were a (necessarily negligible) coupling term to be added the stress-strain relations. The best result, to date, is that this supposition leads to negligible errors, in a mean square sense, for shells with midsurfaces of constant mean curvature[12].

As an alternative to these essentially physical arguments, we shall now show that the validity of the approximation of Clark and others follows from the structure of the Reissner-Meissner equations themselves.

We assume that (2.15), (2.16) and (2.18), (2.19) have unique solutions, each taking on the same prescribed values of β and ψ at $\eta = 0$ and $\eta = l$. Then the differences

$$\bar{\beta} = \beta - \beta_*, \bar{\psi} = \psi - \psi_* \tag{3.1}$$

satisfy the nonhomogeneous equations

$$\epsilon^2[L + (1-\sigma)\phi' \sin\phi]\bar{\beta} - \bar{\psi}\sin\phi = -\epsilon^2(1-\sigma)\phi'\beta_* \sin\phi \tag{3.2}$$

$$\epsilon^2[L + (1+\nu)\phi' \sin\phi]\bar{\psi} + \bar{\beta}\sin\phi = -\epsilon^2(1+\nu)\phi'(\psi_* \sin\phi - \rho v \cos\phi) \tag{3.3}$$

and the homogeneous boundary conditions

$$\bar{\beta}(0) = \bar{\beta}(l) = \bar{\psi}(0) = \bar{\psi}(l) = 0. \tag{3.4}$$

First, we obtain energy estimates for $\bar{\beta}$ and $\bar{\psi}$. To this end we define, for any two pairs of integrable functions (f, g) and $(\bar{f}, \bar{g})$, the (dimensionless) *energy inner product*

$$E_\sigma[f, g; \bar{f}, \bar{g}] = \int_0^l [f\bar{f} + \sigma(f\bar{g} + \bar{f}g) + g\bar{g}]\rho \, d\eta \tag{3.5}$$

and let

$$\|f, g\|_\sigma = \{E_\sigma[f, g; f, g]\}^{1/2}, \ |\sigma| \le 1 \tag{3.6}$$

denote the (dimensionless) energy norm. Note that if $|\sigma| = 1$, the energy norm is only semi-definite.

Multiply (3.2) by $\bar{\beta}$, (3.3) by $\bar{\psi}$, add and integrate from $\eta = 0$ to $\eta = l$. Upon integration by parts and application of the boundary conditions (3.4), the left-hand side of the resulting equation reads

$$-\epsilon^2[\|\bar{\beta}', (\rho'/\rho)\bar{\beta}\|_\sigma^2 + \|\bar{\psi}', (\rho'/\rho)\bar{\psi}\|_{-\nu}^2] \equiv -\epsilon^2(a^2 + b^2). \tag{3.7}$$

(Here and henceforth, we write ρ' in place of $\cos\phi$ for conciseness.)

To evaluate the right-hand side, we use (2.18) to replace $\psi_* \sin\phi - \rho v \cos\phi$ by $\epsilon^2 L\beta_*$ and note that if we solve (2.19) for $-\beta_* \sin\phi$, the resulting equation can be cast into precisely the same form as the left-hand side of (2.13), with ψ replaced by ψ_* and ν replaced by -1. Integration by parts and observation of (3.4) then yields

$$\begin{aligned}&\epsilon^4\{(1+\nu)E_1[\beta_*', (\rho'/\rho)\beta_*; \ (\phi'\bar{\psi})', (\rho'/\rho)\phi\bar{\psi}] \\ &\quad -(1-\sigma)E_1[\psi_*' + p\rho \sin\phi, (\rho'/\rho)\psi_* + v\sin\phi; \ (\phi'\bar{\beta})', (\rho'/\rho)\phi'\bar{\beta}]\}.\end{aligned} \tag{3.8}$$

Let

$$\|f\| = \left(\int_0^l f^2 \rho \, d\eta\right)^{1/2} \tag{3.9}$$

denote the weighted, L_2-norm of any function f and set

$$\kappa = \max_{0 \le \eta \le l} |\phi'(\eta)|. \tag{3.10}$$

Note that κ, the maximum (dimensionless) meridional curvature, must be finite for shell theory to apply. By the bilinearity of E_σ, the inequality

$$(1+\sigma)\|f, g\|_1 \le 2\|f, g\|_\sigma, \ -1 < \sigma, \tag{3.11}$$

Schwarz' inequality

$$|E_\sigma[f, g; \bar{f}, \bar{g}]| \le \|f, g\|_\sigma \cdot \|\bar{f}, \bar{g}\|_\sigma, |\sigma| \le 1, \tag{3.12}$$

and its special case

$$|E_\sigma[f, g; \bar{f}, 0]| \le \|f, g\|_\sigma \cdot \|\bar{f}\|, \tag{3.13}$$

it follows that (3.8), in magnitude, is less than or equal to

$$\begin{aligned}&4\epsilon^2(1-\nu)^{-1}(1+\sigma)^{-1}\{(1+\nu)\|\beta'_*,\ (\rho'/\rho)\beta_*\|_\sigma\\&\times\left[\kappa\|\bar{\psi}',(\rho'/\rho)\bar{\psi}\|_{-\nu}+\frac{1}{2}(1-\nu)\|\phi''\bar{\psi}\|\right]\\&+(1-\sigma)\|\psi'_*+p\rho\sin\phi,(\rho'/\rho)\psi_*+v\sin\phi\|_{-\nu}\\&\times\left[\kappa\|\bar{\beta}',\ (\rho'/\rho)\bar{\beta}\|_\sigma+\frac{1}{2}(1+\sigma)\|\phi''\bar{\beta}\|\right]\Big\}.\end{aligned}\tag{3.14}$$

To further reduce (3.14), let

$$\lambda_\sigma=\max\frac{(1+\sigma)\|\phi''\bar{f}\|}{2\|\bar{f}',\ (\rho'/\rho)\bar{f}\|_\sigma},\tag{3.15}$$

where the maximum is taken over all non-zero functions satisfying $\bar{f}(0)=\bar{f}(l)=0$. ($\lambda_\sigma=0$ for cylindrical, conical, spherical or toroidal shells.) Equating the absolute values of (3.7) and (3.8), dividing by $\epsilon^2\sqrt{(a^2+b^2)}$ and using (3.14) and (3.15) together with the inequalities

$$a,b\le\frac{a^2+b^2}{\sqrt{(a^2+b^2)}},\ \frac{a,b}{\sqrt{(a^2+b^2)}}\le 1,\tag{3.16}$$

we arrive at

$$\begin{aligned}&\|\bar{\beta}',(\rho'/\rho)\bar{\beta}\|_\sigma,\ \|\bar{\psi}',(\rho'/\rho)\bar{\psi}\|_{-\nu}\\&\le 4\epsilon^2(1-\nu)^{-1}(1+\sigma)^{-1}[(1+\nu)(\kappa+\lambda_{-\nu})\|\beta'_*,\ (\rho'/\rho)\beta_*\|_\sigma\\&+(1-\sigma)(\kappa+\lambda_\sigma)\|\psi'_*+p\rho\sin\phi,(\rho'/\rho)\psi_*+v\sin\phi\|_{-\nu}\\&\le\epsilon^2KU_*^{1/2},\end{aligned}\tag{3.17}$$

where K is a constant depending on σ, ν and l, and U_* is the (dimensionless) strain energy functional of the Reissner-Meissner theory evaluated at the solution of the simplified Reissner-Meissner equations.

Pointwise estimates for $\bar{\beta}$ follow upon first observing that $\bar{\beta}(0)=\bar{\beta}(l)=0$ implies

$$|\rho(\eta)\bar{\beta}(\eta)|=\left|\int_0^\eta[\rho(t)\bar{\beta}(t)]'\,dt\right|,\ \left|\int_\eta^l[\rho(t)\bar{\beta}(t)]'\,dt\right|\le\rho(\eta)M(\eta)\cdot\|\bar{\beta}',\ (\rho'/\rho)\bar{\beta}\|_0,\tag{3.18}$$

where

$$\rho(\eta)M(\eta)=\min\left\{\left[\int_0^\eta\rho(t)\,dt\right]^{1/2},\ \left[\int_\eta^l\rho(t)\,dt\right]^{1/2}\right\}.\tag{3.19}$$

$M(\eta)$ will be bounded so long as the meridian does not meet the shell axis in a cusp. Now

$$(1-\sigma^2)\|\bar{\beta}',\ (\rho'/\rho)\bar{\beta}\|_0\le\|\beta',\ (\rho'/\rho)\beta\|_\sigma.\tag{3.20}$$

Hence, by (3.17),

$$|\bar{\beta}(\eta)|\le\frac{\epsilon^2KM(\eta)U_*^{1/2}}{1-\sigma^2}.\tag{3.21}$$

A similar analysis shows that

$$|\bar{\psi}(\eta)|\le\frac{\epsilon^2KM(\eta)U_*^{1/2}}{1-\nu^2},\tag{3.22}$$

The inequality (1.1) follows upon assuming $0\leqq\sigma,\nu\leqq\frac{1}{2}$ and setting $N=\frac{4}{3}KM$.

REFERENCES

1. H. Reissner, *Spannungen in Kugelschalen (Kuppeln)*. pp. 181–193. Muller-Breslau Festschrift, Leipzig (1912).
2. E. Meissner, Das Elastizitätsproblem für dünne Schalen von Ringflächen-Kugel-und Kegel form. *Phys. Zeitschrift* **14**, 343 (1913).
3. E. Reissner, On the theory of thin elastic shells. *H. Reissner Anniversary Volume.* pp. 231–247. J. W. Edwards, Ann Arbor (1949).
4. E. Reissner, On axisymmetrical deformations of thin shells of revolution. *Proc. Sym. Appl. Math.* **3**, 27 (1950).
5. P. M. Naghdi and C. N. DeSilva, On the deformation of elastic shells of revolution. *Q. Appl. Math.* **12**, 369 (1954).
6. B. R. Baker and G. B. Cline, Influence coefficients for thin smooth shells of revolution subjected to symmetric loads. *J. Appl. Mechs* **29**, 335 (1962).
7. C. R. Steele and R. F. Hartung, Symmetric loading of orthotropic shells of revolution. *J. Appl. Mechs.* **32**, 337 (1965).
8. R. A. Clark, Asymptotic solutions of elastic shell problems *Asymptotic Sols. of Diff. Eqs. and Their Appls.* pp. 285–209. Wiley, New York (1964).
9. J. G. Simmonds, Simplification and reduction of the Sanders-Koiter linear shell equations for various midsurface geometries. *Q. Appl. Math.* **28**, 259 (1970).
10. D. A. Danielson, Simplified intrinsic equations for arbitrary elastic shells. *Int. J. Engng. Sci.* **8**, 251 (1970).
11. E. Reissner, On the equations for finite symmetrical deflections of thin shells of revolution. *Progress in Appl. Mechs.* (The Prager Anniversary Volume), pp. 171–178. Macmillan, New York (1963).
12. J. G. Simmonds, Extension of Koiter's L_2-error estimate to approximate shell solutions with no strain energy functional. *Z.A.M.P.* **22**, 339 (1971).